MIT 18.211: COMBINATORIAL ANALYSIS

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Lecture 29: Bipartite Graphs

Bipartite Graphs. In this section, we briefly discuss bipartite graphs. First, we characterize them. For a given bipartite graph, we provide a bound for the size of its set of edges.

Definition 1. Let G be a simple graph. We say that G is *bipartite* if $V(G) = X \cup Y$ for some disjoint sets of vertices X and Y such that every edge of G connects a vertex of X with a vertex of Y.

With notation as in the previous definition, we say that G is a bipartite graph on the *parts* X and Y. The parts of a bipartite graph are often called *color classes*; this terminology will be justified in coming lectures when we generalize bipartite graphs in our discussion of graph coloring.

Example 2. For $m, n \in \mathbb{N}$, the graph G with

$$V(G) = [m+n] \quad \text{and} \quad E(G) = \{ij \mid i \in [m] \text{ and } j \in [m+n] \setminus [m]\}$$

is clearly a bipartite graph on the (disjoint) parts $[m]$ and $[m+n] \setminus [m]$. This graph is called the *complete bipartite graph* on the parts $[m]$ and $[m+n] \setminus [m]$, and it is denoted by $K_{m,n}$.

Example 3. Let C_n by the cyclic graph of length n. Suppose that n is even and write $n = 2k$ for some $k \in \mathbb{N}$ with $k \ge 2$. Labeling the vertices of C_n by $1, 2, \dots, 2k$ so that $v_1 v_2 \dots v_{2k} v_1$ is the cycle of C_n, one can see that $X = \{1, 3, \dots, 2k-1\}$ and $Y = \{2, 4, \dots, 2k\}$ partition $V(G)$ in such a way that C_n is a bipartite graph on the parts X and Y. As a consequence of our next result, C_n is not bipartite when n is odd.

We proceed to characterize bipartite graphs.

Theorem 4. *For a simple connected graph G, the following conditions are equivalent.*

(a) *G is bipartite.*

(b) *Every cycle of G (if some) has even length.*

Proof. (a) $\Rightarrow$ (b): Assume that G is bipartite on the parts X and Y. Suppose, by way of contradiction, that G has a cycle of odd length, namely, $C := v_1v_2 \dots v_{2n+1}v_1$. We can assume, without loss of generality, that $v_1 \in X$. Since no two vertices of X are adjacent, $v_2 \in Y$, and a straightforward inductive argument shows that $v_{2k+1} \in X$ for every $k \in [\![0, n]\!]$ and $v_{2k} \in Y$ for every $k \in [n]$. However, the fact that both v_1 and v_{2n+1} belong to X contradicts that no two vertices in A are adjacent.

(b) $\Rightarrow$ (a): Now suppose that every cycle of G (if some) has even length. Fix $v_0 \in V(G)$, and set the color of v_0 to red. For every $v \in V(G) \setminus \{v_0\}$, set its color to red (resp., blue) if the distance from v_0 to v is even (resp., odd). The distance between two vertices of G is the length of a minimum-length path connecting them. Let R and B be the sets of red and blue vertices of G, respectively. We claim that G is bipartite on the parts R and B. Suppose, towards a contradiction, that this is not the case. Assume that there are two vertices $v, w \in R$ such that $vw \in E(G)$. Let $P_v := v_0v_1v_2 \dots v_{k-1}v$ be a minimum-length path from v_0 to v and let $P_w := v_0w_1w_2 \dots w_{\ell-1}w$ be a minimum-length path from v_0 to w. Then $P := v_0v_1v_2 \dots v_{k-1}vww_{\ell-1} \dots w_2w_1v_0$ is a closed walk in G of odd length because k and ℓ have the same parity (they are both even) and, besides the edges in these two paths, P contains the edge vw. Observe now that if we drop from P all the edges that belong simultaneously to the paths P_v and P_w, then we obtain finitely many edge-disjoint cycles whose lengths add up to an odd number. Hence one of such cycles must have odd length, which is a contradiction. If we assume that two vertices of B are adjacent, then we can arrive to a contradiction in a similar manner. □

Corollary 5. *Every forest is bipartite.*

Unlike trees, the number of edges of a bipartite graph is not completely determined by the number of vertices. In fact, the number of edges is not even determined by the sizes of the two color classes (unless the bipartite graph is complete). However, we can find a tight upper bound for the number of edges in terms of the number of vertices.

Proposition 6. *Let G be a simple bipartite graph on n vertices. Then $|E(G)| \le \frac{n^2}{4}$ if n is even and $|E(G)| \le \frac{n^2-1}{4}$ if n is odd.*

Proof. Suppose that G is bipartite on the parts X and Y, and set $x = |X|$ and $y = |Y|$. It is clear that $|E(G)| \le xy = x(n - x)$. Therefore $E(G)$ is at most

$$M_n := \max\{xn - x^2 \mid x \in [\![0, n]\!]\}.$$

However, $xn - x^2$ is a concave-down parabola with vertex at $x = n/2$. So if $n = 2k$ for some $k \in \mathbb{N}_0$, then

$$M_n = M_{2k} = \frac{2k}{2}(2k - \frac{2k}{2}) = k^2 = \frac{n^2}{4}.$$

On the other hand, if $n = 2k+1$ for some $k \in \mathbb{N}_0$, then

$$M_n = M_{2k+1} = \left\lfloor \frac{2k+1}{2} \right\rfloor \left(2k+1 - \left\lfloor \frac{2k+1}{2} \right\rfloor \right) = k(k+1) = \frac{n-1}{2} \cdot \frac{n+1}{2} = \frac{n^2-1}{4},$$

which concludes the proof. □

Practice Exercises

Exercise 1. [1, Exercise 11.3] *In a round robin tournament there are $2n$ players. Prove that, after the completion of the second round, we can split the competitors into two groups of size n each so that no two competitors in the same group have played with each other yet.*

Exercise 2. *Take a standard chessboard and remove the top left square and the bottom right square. Prove that we cannot cover (without overlapping) the given board using 1×2 dominoes.*

References

[1] M. Bóna: *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory* (Fourth Edition), World Scientific, New Jersey, 2017.

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